Unified spin gauge theory of electroweak and gravitational interactions

1989 J. Phys. A: Math. Gen. 22 1059

(http://iopscience.iop.org/0305-4470/22/8/020)

J. Phys. A: Math. Gen. **22** (1989) 1059-1071. Printed in the UK

Unified spin gauge theory of electroweak and gravitational interactions

J S R Chisholm† and R S Farwell‡

† Mathematical Institute, University of Kent, Canterbury, Kent, UK
‡ St. Mary's College, Twickenham TW1 4SX, Middlesex, UK

Received 1 June 1988

Abstract. A spin gauge theory describing fundamental fermions and their electroweak and gravitational interactions is proposed. It is modelled on an eight-dimensional curved manifold M and uses its associated Clifford algebra. The elements of the algebra are represented by 16×16 matrices and the fermions are represented by sixteen-component column vectors. The frame field is introduced as a result of factorising the fermion mass term in the Lagrangian density and is included in an extended covariant derivative. The usual gauge theoretic technique of defining free bosonic Lagrangians from the fermion covariant derivative, when applied to the extended covariant derivative, gives the correct mass matrix for the photon, W and Z bosons, together with the Einstein-Hilbert gravitational Lagrangian density modified at short distances by a term quadratic in the curvature coefficients. Transformation of the Lagrangian by an inner automorphism of the Clifford algebra gives the correct mass and interaction terms for the up and down quarks.

1. Introduction

In two previous papers [1, 2], we have introduced models of electroweak and gravitational interactions, using the closely related concepts of spin gauge theories and the frame field. These two models were based on Clifford algebras associated with different manifolds, and the gauge groups inducing the two interactions were different. In the present paper, we show that we can unify these two models, using a suitable eight-dimensional manifold M and demanding invariance under the combined gauge group.

The curved manifold M is, as usual, the union of a countable number of patches, on each of which the system of coordinates $\{x^\mu;\ \mu=1,\ldots,8\}$ is non-degenerate. However, in this paper we shall assume that no field quantity is dependent on the coordinates x_5, x_6, x_7, x_8. The derivatives with respect to the coordinates are $\partial_\mu=\partial/\partial x^\mu$, so our assumption essentially implies that the operators ∂_5, ∂_6, ∂_7, ∂_8 are zero.

At any point x of M, the tangent space $T(x)$ is spanned by the vectors $\{\Gamma_i;\ i=1,\ldots,8\}$ of a Clifford algebra, so that

$$\{\Gamma_i,\Gamma_j\}=2Ig_{ij} \tag{1.1}$$

where $\{g_{ij}\}$ is the diagonal metric on the flat space $T(x)$ and I is the unit scalar of the algebra. A fundamental concept of spin gauge theories is that the $\{\Gamma_i\}$ generally becomes x dependent under gauge transformations, although the tangent space metric remains invariant.

The 'frame field' on M is, apart from a constant factor, the set of x-dependent matrices

$$\Gamma_\mu(x) = h^i_\mu(x)\Gamma_i \qquad \mu = 1, \ldots, 8 \tag{1.2}$$

where $\{h^i_\mu(x)\}$ defines the vierbein field on M. The frame field plays an important role in our theories:

(i) it is the square root of the metric $(g_{\mu\nu}(x))$ on M, since

$$\{\Gamma_\mu, \Gamma_\nu\} = 2Ig_{\mu\nu}(x) \tag{1.3}$$

where

$$g_{\mu\nu}(x) = g_{ij}h^i_\mu(x)h^j_\nu(x) \tag{1.4}$$

(ii) it forms a vector basis at each point of M.

The relation (1.2) can be inverted to give

$$\Gamma_i = h_i^\mu(x)\Gamma_\mu(x) \tag{1.5}$$

where

$$h^i_\mu(x)h_i^\nu(x) = \delta^\nu_\mu.$$

In this paper we assume that the 8×8 matrix (h^i_μ) has a 4×4 block diagonal form given by

$$\begin{pmatrix} (h^i_\mu) & 0 \\ 0 & (\delta^i_\mu) \end{pmatrix} \tag{1.6}$$

where $(\delta^i_\mu$: $i, \mu = 5, 6, 7, 8)$ denotes the 4×4 identity matrix. As a consequence of (1.6), we deduce from (1.4) that the metric $(g_{\mu\nu})$ also has block diagonal form. Then the manifold M behaves in a certain sense like the union of two disjoint submanifolds M_1 and M_2. M_1 is a curved four-dimensional submanifold representing spacetime and M_2 is a flat four-dimensional submanifold referred to as the 'higher-dimensional space'. However, since the basis vectors of M_1 and M_2 *anticommute*, M is not the direct product of M_1 and M_2. Thus M_2 cannot be considered to be the 'internal space' of standard gauge theories. This is a crucial difference between spin and standard gauge theories.

The tangent space $T(x)$ can also be considered to be the union of two flat spaces $T_1(x)$ and $T_2(x)$, where $T_2(x)$ is the isomorphic to the flat space M_2. Since we consider M_1 to be spacetime, the tangent space $T_1(x)$ is spanned by the vectors $\{\Gamma_i;\ i = 1, \ldots, 4\}$ of a Dirac algebra, so that

$$\{\Gamma_i, \Gamma_j\} = 2I\eta_{ij} \tag{1.7}$$

where $(\eta_{ij}) = \mathrm{diag}(-1, -1, -1, 1)$ is the Minkowski metric on $T_1(x)$.

Because of the block structure of the metric and vierbein matrices, equation (1.4) is valid for the separate ranges $1, \ldots, 4$ and $5, \ldots, 8$ of all the indices. For the range $5, \ldots, 8$ the equation is trivial. For the ranges $i, j, \mu, \nu = 1, \ldots, 4$, taking the determinant of each side of (1.4) and using the determinant equation

$$\Delta(\eta_{ij}) = -1$$

gives

$$h^2 = -g$$

where $h=\Delta(h^i_\mu)$ and $g=\Delta(g_{\mu\nu})$, so that

$$h=(-g)^{1/2}. \tag{1.8}$$

The invariant 4-volume element on M_1 is thus

$$(-g)^{1/2}\,\mathrm{d}^4x=h\,\mathrm{d}^4x. \tag{1.9}$$

We shall use a 4×4 matrix representation $\{\gamma_i\,;\,i=1,2,3,4\}$ of the Dirac algebra to define a representation of the basis vectors $\{\Gamma_i\,;\,i=1,2,3,4\}$. The pseudoscalar of the Dirac algebra on $T_1(x)$ is

$$\eta=\prod_{i=1}^{4}\gamma_i \tag{1.10}$$

where, using (1.1),

$$\eta^2=-I.$$

Then, by (1.2) and (1.10),

$$\gamma_5=\prod_{\mu=1}^{4}\gamma_\mu=h\eta \tag{1.11a}$$

and if $\gamma^\mu=g^{\mu\nu}\gamma_\nu$, then

$$\gamma^5=\prod_{\mu=1}^{4}\gamma^\mu=-h^{-1}\eta. \tag{1.11b}$$

The eightfold vector basis of T is represented, as in [1], in terms of the Dirac algebra and two sets of 2×2 Pauli matrices $\{\rho_r\}$ and $\{\lambda_s\}$, with ρ_4 and λ_4 equal to the corresponding unit matrices. The basis of the algebra $C(2,6)$ on T is essentially the same as that given by (2.6) of [1]:

$$\Gamma_i=\lambda_4\rho_1\gamma_i \qquad i=1,2,3,4 \tag{1.12a}$$

$$\Gamma_5=-\mathrm{i}\lambda_2\rho_2 I \tag{1.12b}$$

$$\Gamma_6=\mathrm{i}\lambda_1\rho_2 I \tag{1.12c}$$

$$\Gamma_7=\lambda_4\rho_1\eta \tag{1.12d}$$

$$\Gamma_8=\lambda_3\rho_2 I. \tag{1.12e}$$

Again, $\Gamma_4^2=\Gamma_8^2=I$, while $\Gamma_i^2=-I$ $(i\neq 4,8)$. The anticommutation of these eight basis vectors with each other justifies our statement that T is not simply the cartesian product of T_1 and T_2, but a unified vector space spanned by the vectors (1.12).

In § 2 we restrict the gravitational spin gauge transformations to those that keep constant the matrices $\{\lambda_r, \rho_s\,;\,r,s=1,2,3,4\}$ and the Dirac pseudoscalar η. Thus the basis vectors $\{\Gamma_i\,;\,i=5,6,7,8\}$ of T_2 remain constant under the gravitational spin gauge transformations. The constancy of $\{\lambda_r,\rho_s\}$ also implies that $\{\Gamma_\mu\,;\,\mu=5,6,7,8\}$ is essentially x dependent through the action of the set $\{h^i_\mu\}$ on the vectors of the Dirac algebra. It is characteristic of spin gauge theories that $\{\Gamma_\mu\}$ obtains further x dependence through the other spin gauge transformations. The pseudoscalar η remains constant under all gauge transformations, which ensures that the meaning of helicity states, for instance

$$\varepsilon_{\mathrm{L}}=\tfrac{1}{2}(I+\mathrm{i}\eta)\varepsilon \qquad \varepsilon_{\mathrm{R}}=\tfrac{1}{2}(I-\mathrm{i}\eta)\varepsilon \qquad \nu_{\mathrm{L}}=\tfrac{1}{2}(I+\mathrm{i}\eta)\nu \qquad \nu_{\mathrm{R}}=\tfrac{1}{2}(I-\mathrm{i}\eta)\varepsilon \tag{1.13}$$

for electrons and neutrino spinors, is invariant under gauge transformations.

As in (2.2) in [1], we take the sixteen-component state vector to be

$$\psi = [\varepsilon_L \varepsilon_R \nu_L \nu_R]^T \tag{1.14}$$

with the matrices $\{\rho_s\}$ acting on the top pair and the bottom pair of spinors, while $\{\lambda_r\}$ act on the different particle blocks in (1.14). The helicity projection operators in the tangent space $T(x)$ are

$$h_\pm = \tfrac{1}{2}(I_{16} \pm i\lambda_4\rho_4\eta). \tag{1.15}$$

As in § 4 of [1],we obtain the electroweak interactions of the quarks by changing the representation of the elements of the Clifford algebra associated with the tangent space $T(x)$. The representation is changed by applying an inner automorphism to the vector basis

$$\Gamma_i \rightarrow T_\alpha \Gamma_i T_\alpha^{-1} \tag{1.16}$$

where

$$T_\alpha = \exp[-i\lambda_3\rho_1 I\alpha] \tag{1.17}$$

and

$$\cos 2\alpha = -\tfrac{1}{3}. \tag{1.18}$$

We adopt the same principle as in [1] that the sixteen-component quark state vector has the same form as the lepton state vector (1.15). As before, our model is limited by the inclusion of only one colour of quark, and we make no attempt to include strong interactions or more than one generation.

2. Gauge symmetries and interactions

The two sets of gauge transformations generating the gravitational and electroweak interactions are essentially those given in § 2 of [2] and in § 2 of [1] respectively. For leptons, the sixteen-component state vectors are given by (1.14) and the kinetic term in the Lagrangian density is given at the beginning of § 2 of [1], with γ_5 replaced by η in (2.3) and (2.7) of [1]. As in (2.9) of [1], the generators of the SU(2) and U(1) symmetry groups are

$$U_1 = \tfrac{1}{2}i\Gamma_6\Gamma_7 = \tfrac{1}{2}i\lambda_1\rho_3\eta \tag{2.1a}$$

$$U_2 = \tfrac{1}{2}i\Gamma_7\Gamma_5 = \tfrac{1}{2}i\lambda_2\rho_3\eta \tag{2.1b}$$

$$U_3 = \tfrac{1}{2}\Gamma_5\Gamma_6 = \tfrac{1}{2}\lambda_3\rho_4 I \tag{2.1c}$$

and

$$P = \tfrac{1}{2}i\Gamma_5\Gamma_6\Gamma_7\Gamma_8 = \tfrac{1}{2}i\lambda_4\rho_3\eta. \tag{2.1d}$$

The spin gauge transformation generating the lepton electroweak interactions is, as in (2.11) of [1],

$$Q(x) = \exp[-igh_+ U_a\theta_a(x) - ig'(h_- U_3 + P)\theta_4(x)] \tag{2.2}$$

with summation over $a = 1, 2, 3$. The state vector (1.14) and the tangent space vectors $\{\Gamma_i\,;\, i = 1, \ldots, 8\}$ are transformed under the electroweak spin gauge transformations by

$$\psi \rightarrow Q\psi \qquad \Gamma_i \rightarrow Q\Gamma_i Q^{-1}.$$

Invariance of the lepton Lagrangian under these transformations is ensured by introducing the covariant derivative

$$D_{\mu\mathrm{EW}} = \partial_\mu - \Omega_\mu \tag{2.3}$$

where the electroweak spin connection is given by

$$\begin{aligned}\Omega_\mu &= \mathrm{i}gh_+ U_a W_{a\mu} + \mathrm{i}g'(h_- U_3 + P) W_{4\mu} \\ &= \tfrac{1}{2}\{\mathrm{i}gh_+(\mathrm{i}\lambda_j\rho_3\gamma_5 W_{j\mu} + \lambda_3\rho_4 I W_{3\mu}) + \mathrm{i}g'(h_-\lambda_3\rho_4 I + \mathrm{i}\lambda_4\rho_3\eta) W_{4\mu}\}\end{aligned} \tag{2.4}$$

with summation over $j = 1, 2$. Under the electroweak gauge transformations, the spin connection transforms according to

$$\Omega_\mu \to Q\Omega_\mu Q^{-1} - Q(\partial_\mu Q^{-1}) \tag{2.5}$$

which is equivalent to the usual SU(2) and U(1) transformations on $\{W_{a\mu}\,;\, a = 1, 2, 3\}$ and $W_{4\mu}$. As in [1], after reduction of the sixteen-component lepton Lagrangian to four-component form and rotation through the Weinberg angle, Ω_μ gives the usual electroweak interactions for the electron–neutrino system. The transformation of the lepton Lagrangian density to the 'quark representation' through the inner automorphism (1.16)-(1.18) produces the correct electromagnetic and weak interactions of the up and down quarks. The lepton Lagrangian kinetic term is invariant under the transformation (1.16) and (1.17) and so is also correct for quarks.

To introduce the gravitational field, let us consider the generalisation of the covariant derivative introduced in § 1 of [2]. If I_{16} denotes the (16×16) unit matrix, the generalisation of (1.10) of [2] is

$$D_{\mu\mathrm{G}} = I_{16}\partial_\mu - G_\mu \tag{2.6}$$

and the covariant derivatives of the basis vectors $\{\Gamma_i\}$ are

$$D_{\mu\mathrm{G}}\Gamma_i = \partial_\mu\Gamma_i - \Gamma^j_{\mu i}\Gamma_j - [G_\mu, \Gamma_i] \tag{2.7}$$

where $\{\Gamma^j_{\mu i}\}$ is the Christoffel vector connection. We note that the suffixes i, j range over the values $i, j = 1, 2, \ldots, 8$, but, since M_2 is flat, it follows that

$$\Gamma^j_{\mu i} = 0 \qquad i = 5, 6, 7, 8 \text{ and/or } j = 5, 6, 7, 8. \tag{2.8}$$

In place of (1.15) of [2] we assume that there is a gauge in which the eight parallel transport conditions:

$$D_{\mu\mathrm{G}}\Gamma_i = 0 \qquad i = 1, 2, \ldots, 8 \tag{2.9}$$

hold. We are also assuming that $\{\lambda_r\}$, $\{\rho_s\}$ and η are kept constant under the gravitational spin gauge transformations, so that, from (1.12),

$$\partial_\mu\Gamma_i = 0 \qquad i = 5, 6, 7, 8. \tag{2.10}$$

Equations (2.7)-(2.10) imply that

$$[G_\mu, \Gamma_i] = 0 \qquad i = 5, 6, 7, 8 \tag{2.11}$$

and we deduce that G_μ must be constructed from I, λ_4, ρ_4 and the four Dirac basis vectors γ_1, γ_2, γ_3, γ_4 only. Then, as in § 1 of [2], the first four equations (2.9) imply that

$$G_\mu = G^{ij}_\mu \lambda_4\rho_4\gamma_{ij} \qquad i, j = 1, 2, 3, 4 \tag{2.12a}$$

where

$$G^{ij}_\mu = \tfrac{1}{2} g^{\alpha\beta} h^j_\beta (D_{\mu\mathrm{G}} h^i_\alpha) \qquad \alpha, \beta = 1, 2, 3, 4 \tag{2.12b}$$

and we have again chosen the arbitrary scalar component of G_μ to be zero. So the connection G_μ differs from its value in [2] only by the extra unit matrix factor $\lambda_4\rho_4$. Therefore equations (1.20)-(1.24) of [2] hold with only minor modifications; most importantly, the generalisations of (1.23) and (1.24) of [2] are

$$\lambda_4\rho_4(\partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu) = [G_\mu, \lambda_4\rho_4\gamma_\nu] - [G_\nu, \lambda_4\rho_4\gamma_\mu] \tag{2.13a}$$

and

$$\mathrm{Tr}\{g^{\mu\rho}g^{\nu\tau}G_{\mu\nu}\lambda_4\rho_4[\gamma_\rho, \gamma_\tau]\} = -16R \tag{2.13b}$$

where

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + [G_\mu, G_\nu] \tag{2.13c}$$

and R is the curvature scalar of the spacetime manifold M_1.

Also, the gauge transformations under which $D_{\mu G}$ is covariant are, as in (1.18) and (1.19) of [2],

$$S = \exp(\theta^{ij}\lambda_4\rho_4\gamma_{ij}) \tag{2.14}$$

with

$$G_\mu \to SG_\mu S^{-1} - S(\partial_\mu S^{-1}). \tag{2.15}$$

The transformation S represents a local Lorentz transformation in the spin space associated with spacetime M_1.

The lepton electroweak generators $\{U_a\}$ and P are respectively bivectors and the quadrivector formed from $\{\Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8\}$, and so commute with the bivectors $\lambda_4\rho_4\gamma_{ij}$ in S and G_μ. Since η also commutes with G_μ, it follows that the electroweak transformation Q and spin connection Ω_μ commute with the corresponding gravitational quantities S and G_μ. So the spin gauge transformation formed from combining the electroweak and the spin Lorentz transformations can be represented by either QS or SQ.

The fact that Q and S commute also implies that the full covariant derivative

$$D_\mu = I_{16}\partial_\mu - G_\mu - \Omega_\mu \tag{2.16}$$

is covariant under the combined transformation; that is,

$$D_\mu \to QSD_\mu(QS)^{-1} \tag{2.17}$$

provided that G_μ and Ω_μ transform by (2.15) and (2.5) respectively. The transformation T_α used in the inner automorphism (1.16)-(1.18) to obtain the quark interactions also commutes with S. Hence our comments concerning the combined transformation also apply in the quark representation.

We noted in [1] that it was possible to use the commutator $[D_{\mu\mathrm{EW}}, D_{\nu\mathrm{EW}}]$ to define the 'curls' of the fields $\{W_{a\mu}\}$ and $W_{4\mu}$, normalising them separately. With D_μ now given by (2.16), we find

$$[D_\mu, D_\nu] = -\mathrm{i}gh_+U_aW_{a\mu\nu} - \mathrm{i}g'(h_-U_3 + P)W_{4\mu\nu} + G_{\mu\nu} \tag{2.18}$$

where

$$W_{a\mu\nu} = \partial_\mu W_{a\nu} - \partial_\nu W_{a\mu} + g\varepsilon_{abc}[W_{b\mu}, W_{c\nu}] \qquad a = 1, 2, 3 \tag{2.19a}$$

$$W_{4\mu\nu} = \partial_\mu W_{4\nu} - \partial_\nu W_{4\mu} \tag{2.19b}$$

and $G_{\mu\nu}$ is the bivector given by (2.13c).

From (2.18) we can project out the curls of the four boson fields as the gauge-invariant traces:

$$\mathrm{i}\,\mathrm{Tr}\{h_+ U_a[D_\mu, D_\nu]\}/4g = \tfrac{1}{2}W_{a\mu\nu}$$

$$\mathrm{i}\,\mathrm{Tr}\{(h_- U_3 + P)[D_\mu, D_\nu]\}/12g' = \tfrac{1}{2}W_{4\mu\nu}.$$

The bivector $G_{\mu\nu}$ is also clearly gauge-covariant. So it is possible to construct, as in (2.26) of [1] and (2.4) of [2], the gauge-invariant and separately normalisable 'free boson' and quadratic curvature Lagrangian terms

$$\tfrac{1}{4}W_{a\mu\nu}W_a^{\mu\nu} \tag{2.20a}$$

$$\tfrac{1}{4}W_{4\mu\nu}W_4^{\mu\nu} \tag{2.20b}$$

and

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2.20c}$$

However, we noted in [1] that, if the Weinberg angle is given by $\theta_{\mathrm{w}} = \pi/6$, the normalisation factors of the terms (2.20a) and (2.20b) are equal. For our present model, in which generation mixing is ignored and strong interactions are not included, this is a good approximation to the measured value. We shall therefore adopt the principle suggested in [1], that the terms (2.20) are all derived from $[D_\mu, D_\nu]$ using a common normalisation factor. Then, in order to obtain the correct normalisations for (2.20a, b), the kinetic part of the boson Lagrangian density must be, as in (3.23) of [1],

$$\begin{aligned} L_{BK} &= (1/16g^2)\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[D_\mu, D_\nu][D_\rho, D_\sigma]\} \\ &= \tfrac{1}{4}W_{a\mu\nu}W^{a\mu\nu} + \tfrac{1}{4}W_{4\mu\nu}W^{4\mu\nu} + (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}/16g^2). \end{aligned} \tag{2.21}$$

This Lagrangian density is invariant under the transformation (1.16)-(1.18) to the quark representation. It is very interesting that the principle that fixes the Weinberg angle also determines the coefficient of the last term in (2.21), which we call the 'spin gravity' Lagrangian term. In (2.12) of [2], the coefficient of this term was quite different, depending upon the mass m of a 'representative fermion' in the model; it is far more satisfactory that the coefficient depends only upon the fine structure constant, as in (2.21). We shall discuss the consequences of this change of normalisation later.

3. Mass, gravitation and the frame field

One of the central ideas of [1, 2] is the rewriting of the mass term in the Dirac equation in terms of the frame field. In order to deal with the masses of both quarks and leptons, we shall introduce Lagrangian mass terms in the form (4.8) of [1]. The four lepton and quark masses $\{m_\varepsilon, m_\nu, m_u, m_d\}$ are related to mass constants $\{\mu_i;\, i = 1, 2, 3, 4\}$ by the equations (4.11)and (4.22) of [1]:

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = m_\varepsilon \tag{3.1a}$$

$$\mu_1 - \mu_2 + \mu_3 - \mu_4 = m_\nu \tag{3.1b}$$

$$\mu_1 + \mu_2 - \tfrac{1}{3}(\mu_3 + \mu_4) = m_u \tag{3.1c}$$

$$\mu_1 - \mu_2 - \tfrac{1}{3}(\mu_3 - \mu_4) = m_d. \tag{3.1d}$$

As in [1] we take the view that fermion mass is an interaction between a fermion and the frame field, factoring out coupling constants $\{f_i\}$ from the mass constants $\{\mu_i\}$; the 'frame inertia' ζ of the frame field then satisfies

$$\mu_i = 4f_i\zeta \qquad i = 1, 2, 3, 4. \tag{3.2}$$

The factor ζ is incorporated into the frame field

$$\phi_\mu(x) = \zeta\gamma_\mu(x) \tag{3.3}$$

and by using the identity $\gamma^\mu(x)\gamma_\mu(x) = 4I$, which holds on any non-singular patch of a manifold, a fermion scalar mass term can be written in the form

$$\mu_i I = f_i\gamma^\mu(x)\phi_\mu(x). \tag{3.4}$$

Then, as in (4.8) of [1], the lepton mass term is

$$\tfrac{1}{2}\bar{\psi}\mathrm{i}\lambda_4\rho_1\gamma_\mu[\mathrm{i}f_1\lambda_4\rho_1\phi_\mu + \mathrm{i}f_2\lambda_3\rho_1\phi_\mu - \mathrm{i}f_3\lambda_4\rho_2\eta\phi_\mu - \mathrm{i}f_4\lambda_3\rho_2\eta\phi_\mu]\psi + \text{conj}. \tag{3.5}$$

Using (3.1*a*, *b*), (3.2) and (3.3), and reducing (3.5) to four-component form, gives the correct electron and neutrino mass terms. As in [1], when (3.5) is transformed to a different representation of the Clifford algebra using (1.16)-(1.18), terms equivalent to the quark mass terms are obtained. Adding the term

$$E_\mu = \mathrm{i}f_1\lambda_4\rho_1\phi_\mu + \mathrm{i}f_2\lambda_3\rho_1\phi_\mu - \mathrm{i}f_3\lambda_4\rho_2\eta\phi_\mu - \mathrm{i}f_4\lambda_3\rho_2\eta\phi_\mu \tag{3.6}$$

to the covariant derivative (2.16) gives the 'extended covariant derivative':

$$\Delta_\mu = D_\mu + E_\mu \tag{3.7}$$

$$= I_{16}\partial_\mu - G_\mu - \Omega_\mu + E_\mu. \tag{3.8}$$

The terms D_μ and E_μ in (3.7) are separately gauge covariant, which implies that the three commutator combinations

$$[D_\mu, D_\nu] \qquad [D_\mu, E_\nu] - [D_\nu, E_\mu] \qquad [E_\mu, E_\nu] \tag{3.9}$$

are also separately gauge covariant and so can be separately normalised. This turns out to be necessary in this model, as it was in the models in [1]. We still conjecture that, in a full theory, the normalisations of terms such as (3.9) should be the same, all contributing to the Lagrangian through a term of the form

$$-K\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[\Delta_\mu, \Delta_\nu][\Delta_\rho, \Delta_\sigma]\} \tag{3.10}$$

where K is an appropriate constant. We have already been able to assume this property for the Lagrangian contributions arising solely from D_μ.

We can regard the trace in (3.10) as made up of traces involving pairs of terms of the form (3.9). If we normalise each of these terms separately, giving

$$N_1[D_\mu, D_\nu] \qquad N_2([D_\mu, E_\nu] - [D_\nu, E_\mu]) \qquad N_3[E_\mu, E_\nu] \tag{3.11}$$

then we can form nine different traces involving pairs of these terms. However, the only element of the (16×16) Clifford algebra which has non-zero trace is I_{16}; this ensures that five of these nine traces are zero. The four non-zero traces are

$$N_1^2\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[D_\mu, D_\nu][D_\rho, D_\sigma]\} \tag{3.12a}$$

$$N_2^2\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}([D_\mu, E_\nu] - [D_\nu, E_\mu])([D_\rho, E_\sigma] - [D_\sigma, E_\rho])\} \tag{3.12b}$$

$$N_3^2\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[E_\mu, E_\nu][E_\rho, E_\sigma]\} \tag{3.12c}$$

and

$$2N_1N_3\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[D_\mu, D_\nu][E_\rho, E_\sigma]\}. \tag{3.12d}$$

Each of these terms is relativistically and gauge invariant, and so can contribute independently to the Lagrangian.

The contribution (3.12*a*) has already been considered in § 2 and contributes the expression (2.21) to the Lagrangian. The normalisation factor in (3.12*a*) thus requires

$$N_1 = 1/4g. \tag{3.13}$$

Using (3.3), and taking the traces of the λ and ρ matrices, the term (3.12*c*) is equal to

$$4N_3^2F^4\zeta^4\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}[\gamma_\mu, \gamma_\nu][\gamma_\rho, \gamma_\sigma]\}$$

where, as in (4.16) of [1],

$$F^2 = \sum f_i^2. \tag{3.14}$$

Since

$$[\gamma_\mu, \gamma_\nu][\gamma_\rho, \gamma_\sigma] = 4(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})I$$

and $\mathrm{Tr}\, I = 4$, (3.12*c*) is equal to

$$\begin{aligned} 64N_3^2F^4\zeta^4g^{\mu\rho}g^{\nu\sigma}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) &= 64N_3^2F^4\zeta^4(\delta^\rho_\sigma\delta^\sigma_\rho - \delta^\rho_\rho\delta^\sigma_\sigma) \\ &= -768N_3^2F^4\zeta^4 \\ &= -3N_3^2M^4 \end{aligned} \tag{3.15}$$

where, as in (4.18) of [1],

$$M^2 = \sum \mu_i^2. \tag{3.16}$$

The essential point about the result (3.15) is that the term is constant for all non-degenerate metrics and in all gauges, and so provides a cosmological constant term in the Lagrangian. We now consider the remaining terms (3.12*b*) and (3.12*d*).

In a gauge satisfying the parallel transport condition (2.9). G_μ takes the spacetime bivector form defined by (2.12) and (2.13). Since $[\gamma_\mu, \gamma_\nu]$ is also a spacetime bivector, only the term $G_{\mu\nu}$ in (2.18) contributes to (3.12*d*). As in the calculation of (2.10) of [2], this contribution is, using (2.13*b*),

$$2N_1N_3F^2\zeta^2\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}G_{\mu\nu}\lambda_4\rho_4[\gamma_\mu, \gamma_\nu]\} = -2N_1N_3M^2R. \tag{3.17}$$

To make this equal to the Einstein–Hilbert free gravitational Lagrangian density

$$L_G = R/16\pi G \tag{3.18}$$

where G is the gravitational constant, we must choose

$$\begin{aligned} N_3 &= -(32\pi GM^2N_1)^{-1} \\ &= -(g/8\pi GM^2) \end{aligned} \tag{3.19}$$

using (3.13). The ratio of the two normalisation constants (3.13) and (3.19) is thus

$$N_1/N_3 = -2\pi GM^2/g^2 \tag{3.20}$$

which is of the order of the ratio of the gravitational to the electrical force between two charged particles. It is, of course, no surprise that this ratio enters into a model which deals with both electroweak and gravitational forces. The result (3.20) demonstrates the extreme impossibility of a common normalisation factor in this particular theory.

To evaluate (3.12*b*), we note first that

$$[D_\mu, E_\nu]-[D_\nu, E_\mu]$$
$$=[I_{16}\partial_\mu, E_\nu]-[I_{16}\partial_\nu, E_\mu]-[G_\mu, E_\nu]+[G_\nu, E_\mu]-[\Omega_\mu, E_\nu]+[\Omega_\nu, E_\mu]$$

and that the first four terms here cancel in a gauge in which (2.9) holds, using the equality (2.13*a*). So (3.12*b*) becomes

$$N_2^2\,\mathrm{Tr}\{g^{\mu\rho}g^{\nu\sigma}([\Omega_\mu, E_\nu]-[\Omega_\nu, E_\mu])([\Omega_\rho, E_\sigma]-[\Omega_\sigma, E_\rho])\}. \tag{3.21}$$

If we choose the general normalisation factor

$$N_2=1/4k \tag{3.22}$$

where k is a constant, then the calculation of (3.21) is exactly as in [1], leading to the result

$$L_M=-\tfrac{1}{2}(Mg/2k)^2g^{\mu\nu}[W_{1\mu}W_{1\nu}+W_{2\mu}W_{2\nu}+\sec^2\theta_W Z_\mu Z_\nu]. \tag{3.23}$$

This calculation depends upon identities similar to those in (3.17) of [1], for example

$$\mathrm{Tr}[g^{\mu\nu}\lambda_2^2\rho_2^2\gamma_\mu\gamma_\nu]=64 \tag{3.24a}$$

and

$$\mathrm{Tr}[g^{\mu\nu}\lambda_1^2\rho_2^2\eta\gamma_\mu\eta\gamma_\nu]=64 \tag{3.24b}$$

which hold *independently of the manifold metric or the choice of gauge. The boson mass terms* (3.23) *thus depend on the gauge only through the gauge dependence of the potentials* $\{W_{i\mu}, Z_\mu\}$. In [1] we introduced the 'Dongpei gauge', in which the $\{\gamma_\mu\}$ are constant. On the curved manifold M this gauge does not exist; the concept is, however, irrelevant, since identities such as (3.24) are universally true through (1.3).

As in (4.19) and (4.20) of [1], the mass of the W boson is identified as

$$M_W=(Mg/2k). \tag{3.25}$$

However, since we do not equate k to F, as in [1], this identification of M_W does not fix the value of the frame inertia ζ; we shall discuss this point further in § 4. As in [1], (3.25) implies that the kinetic and mass normalisation constants N_1 and N_2, given by (3.13) and (3.22), are not equal in this model, since (3.25) can be written

$$N_1/N_2=k/g=M/2M_W\neq 1.$$

But if the model were amended to include fermions with mass of the order of M_W, the equality $N_1=N_2$ could hold. So the hypothesis that the boson Lagrangian will ultimately be of the form (3.10) requires the 'composite fermion mass' M to equal $2M_W$; it is certainly possible that three of four generations of fermions could give this result.

One major difference between the results of this paper and those of [1, 2] is the normalisation of the last term in (2.21), the 'spin gravity' term. We can write the sum of this term and the Einstein-Hilbert term (3.18) as

$$(R+m_0^{-2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})/16\pi G \tag{3.26}$$

where

$$\pi G m_0^2 = g^2. \tag{3.27}$$

So m_0 is of the order of the Planck mass, and the range of the short-range 'spin gravity' force resulting from the quadratic term in (3.26) is the Planck length. This is, in fact, far more satisfactorily than the conclusion in [2], that the range of these forces was of the order of the Compton wavelength of a 'typical fermion'.

Under the transformation (1.16) to the quark representation, the normalised commutators (3.11) are covariant, as we pointed out in § 4 of [1], so that the traces (3.12) are unchanged by the transformation to the quark representation. Therefore all the boson Lagrangian density terms (2.21), (3.18) and (3.23) are invariant under the transformation from lepton to quark representation. This is satisfactory, since the boson field properties should not depend upon the fermion representation.

4. Summary and discussion

In our two previous papers [1, 2], we formulated spin gauge theories of first-generation electroweak interactions and of gravitation. In this paper we have shown that we can, with a few modifications, combine these results, since the generators of the electroweak and gravitational gauge transformations commute. We shall now summarise our assumptions and results.

(i) We use an eight-dimensional manifold M which is a combination of a four-dimensional curved submanifold representing spacetime and a flat 4-dimensional submanifold. The tangent space to M is spanned by eight anticommuting basis vectors $\{\Gamma_i; i = 1, \ldots, 8\}$ of the Clifford algebra C(2, 6).

(ii) The generators of the electroweak gauge transformations are constructed from the basis vectors $\Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8$ and the helicity projection operators. The generators of the gravitational interaction are the (Lorentz) bivectors of spacetime. We use a representation of $\{\Gamma_i; i = 1, 2, 3, 4\}$ which contains the Dirac matrices $\{\gamma_i; i = 1, 2, 3, 4\}$ as a factor. The matrices $\Gamma_\mu(x) = h^i_\mu(x)\,\Gamma_i$ are x dependent through (a) the vierbein field $h^i_\mu(x)$ and (b) the spin gauge transformations. This second dependence is characteristic of spin gauge theories and ensures that $\{\Gamma_\mu(x)\}$ must be regarded as a field. Since $\Gamma_\mu(x)$ are the basis vectors, we call this field the 'frame field'.

(iii) By factorisation of the mass term in the Dirac equation, fermion mass is reinterpreted as an interaction of a fermion with the frame field; so mass is no longer seen as a purely intrinsic property of particles, as in Newtonian mechanics.

(iv) By adding the frame field interaction term to the electroweak and gravitational bivector interaction terms in the 'extended covariant derivative', we obtain Lagrangian density terms corresponding to (a) the electroweak boson mass matrix (with an arbitrary multiplicative constant) and (b) Einstein–Hilbert gravitation.

(v) The standard electroweak and gravitational theories are thereby modified in the following ways: (a) the Higgs–Kibble mechanism is not needed for deriving the boson mass matrix and (b) the quadratic 'spin gravity' term, providing an additional short-range interaction, is added to the usual Einstein–Hilbert Lagrangian density.

(vi) As far as possible, we have adopted the principle of 'equal normalisation' expressed in (3.10). This principle has to be violated for the Einstein–Hilbert term, the familiar factor of order 10^{38} arising from the different normalisations. The remaining 'kinetic' and 'mass' terms in the Lagrangian density can have equal normalisation if (*a*) the Weinberg angle is $\pi/6$ *and* the ratio of the ordinary to spin gravitational terms is of the order of the square of the Planck length and (*b*) the 'combined fermion mass' is $2M_W$, implying that there are fermions in nature whose mass is of the order of M_w.

In this paper, the concept of the 'frame field' plays a central role and occurs in several different physical contexts.

(i) The spacetime components contain the Dirac matrices as factors, so the frame field is related in the usual way to the spin and energy sign of fermion states.

(ii) Fermion mass is seen as an interaction between a fermion and the frame field, with interaction strength proportional to the fermion mass.

(iii) The correct electroweak boson mass matrix is derived by including the frame field interaction in the 'extended covariant derivative'.

(iv) Through the basic Clifford algebra relation (1.3), the frame field is related to the metric and thus, in spacetime, to gravitation.

(v) The inclusion of the frame field term in the extended covariant derivative also leads to the usual gravitational Lagrangian density, so we are proposing a *microscopic* theory of gravitation, with a very close relationship between particle masses and gravity.

(vi) The frame field is seen as a background to other fields, defining the metric. The philosophical concept of 'empty space' appears to be redundant.

(vii) The close relationship of the frame field to gravity is underlined by the fact that the 'free frame field' term in (3.15) of [1] is absorbed when gravitation is included in the model.

It is well known that fermion mass breaks chiral symmetry; it is less well known that the free neutrino equation is the extension of the Cauchy–Reimann equations to spacetime [3], so that a fermion mass term also 'breaks analyticity'. Close examination of our calculations reveals that non-zero boson masses arise because helicity symmetry is broken. So mass is an analytic and algebraic disaster.

Adding the spin gravity term to the Einstein–Hilbert term in (3.28) changes the second-order Einstein equations to fourth-order equations; as we noted in [1], this type of equation has been proposed in order to help the convergence of gravitational theories and to account for inflation. We are interested in using the extra boundary conditions to fit the properties of a fermion source at short distances. References to several papers which discuss these fourth-order equations are given in [1].

Our model has some obvious limitations: it accounts for only one generation of particles, strong interactions and colour are not included, the theory is not quantised and we have not studied renormalisation. The problem of renormalisation is now more complicated than in [1], since gravity and spin gravity have to be included. An apparent arbitrariness of our model is that the electroweak generators have been chosen phenomenologically in order to give the correct lepton interactions. We have noted that these generators happen to satisfy certain conditions essential to the successful formulation of the theory, but we have not asked the converse question: what restrictions on the generators follow from the imposition of these essential conditions?

In order to include gravitation, we have curved the spacetime dimensions. The higher dimensions are, however, flat. An obvious possible development of our model would be to allow the higher dimensions also to be curved, and to introduce Kaluza-Klein ideas; this is one way in which the Planck length might appear more naturally.

One unanswered basic question is whether, in a quantised theory, frame field quanta can be separately observed. The basic relations (1.3) might be interpreted as a statement that a 'graviton' is the symmetric state of two frame field quanta; is this the only combination in which these quanta can be observed? The question of unitarity of the theory also arises, partly because the frame field has zero mass, and so has long range. The long-range forces may, in fact, turn out to be useful in explaining Mach's principle: they might provide a preferential frame of reference relative to the background of stars.

References

[1] Chisholm J S R and Farwell R S 1987 *J. Phys. A: Math. Gen.* **20** 6561
[2] Chisholm J S R and Farwell R S 1988 *Gen. Rel. Grav.* **20** 371
[3] Brackx F, Delanghe R and Sommen F 1982 *Clifford Analysis* (London: Pitman)